Hopfions and Their Kin: A Hopf Index-Based Study of 3D Magnetic Topology

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In 3D systems, the addition of a third spatial dimension enriches the applications of topology by enabling knots, links, and braids, to form textures such as Hopfions [1] and screw dislocations [2]. We show that the conventional classification of 3D magnetic textures by homotopy groups is not sufficient to explain all classes of 3D topological magnetic defects such as magnetic textures in non-uniform backgrounds. Instead, it is required to introduce the concept of flux tubes of the emergent magnetic field $\boldsymbol{F}$ corresponding to the magnetic textures and analyse their linking numbers as topological invariants. We introduce and apply a new formula for calculating the Hopf index H based on the flux-weighted average of these linking numbers [3]

$$H = \sum_i L_{ii}\Phi_i^2 + \sum_{i\neq j} L_{ij}\Phi_i\Phi_j$$

where $L_{ij} = -\frac{1}{4\pi}\oint_{C_i}\oint_{C_j}\frac{\boldsymbol{r}_i-\boldsymbol{r}_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|^3}\cdot\left(d\boldsymbol{r}_i\times d\boldsymbol{r}_j\right)$ is the Gauss linking number of two closed curves belonging to flux tubes labelled by the index i and j.
Here Φ_i is the (positive definite) flux of $\boldsymbol{F}$ through the flux tube i, and L_{ii} and L_{ij} are respectively the self-linking of flux tube i, and inter-linking between flux tubes i and j. This formalism allows for the analytical calculation of H and offers insight into the interpretation of H as a measure of the linkage and topology of the system, while surpassing the pitfalls and limitations associated with calculating the conventional Whitehead integral formula numerically in finite volumes [4].

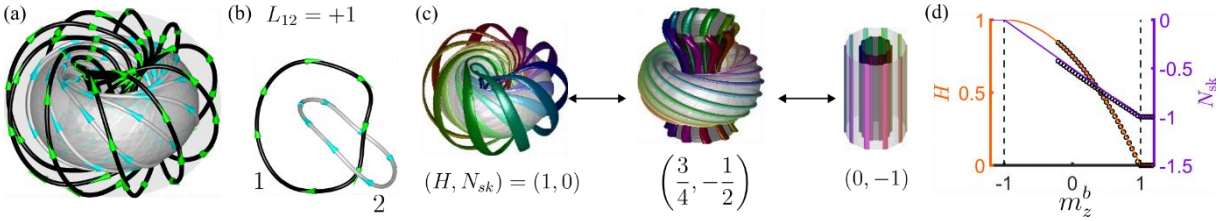

Figure 1. Topological characterization of 3D textures: (a) Field lines of emergent magnetic field $\boldsymbol{F}$ within a Hopfion (b) field lines of $\boldsymbol{F}$ form closed loops that link once with any other field line, (c) conversion of a Hopfion (left) to a Skyrmion tube (right) through an intermediate "mixed topology" state (center) with non-integer N_{sk} and H (d) N_{sk} and H change continuously during the transition shown in (c) and can take non-integer values. The black circles show numerically calculated values, the orange and purple lines show analytical results.

We calculate H for many examples of magnetic textures in ferromagnetic, spiral, or screw dislocation backgrounds [3,4], such as self-linked and inter-linked Skyrmions, meron tubes and Hopfions. We show that magnetic textures with different topological indices can be smoothly transformed into each other and intermediate states along these pathways possess "mixed topology", and can correspond to local energy minima. This framework naturally explains the occurrence of textures with non-integer Hopf index values or fractional Hopfions.

References

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